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# A Clarification of Some Current Multiplicative Confusion Models [1]

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Four recently proposed models of confusion are examined within a classification of major types of multiplicative models. It is shown that all four are equivalent within a single confusion matrix but are open to mutual testing across experiments producing two or more confusion matrices. Generalization of these models is considered.

Consider the class of experimental situations where there are $n$ possible stimulus presentations and $n$ possible responses and there is an experimenter-defined one–one stimulus–response mapping. Then a confusion matrix is a table with the following properties: The stimuli are given row identification such that the cell entries are conditional frequencies that the column response is made to the row stimulus.

More formally, $\{c_{ij}\}$ represents a confusion matrix with positive entries in its cells such that $\sum_{j=1}^{n} c_{ij} = 1$, for $1 \leqslant i \leqslant n$, where $i$ identifies the stimulus and the $j$ the response.

Confusion matrices are useful in the study of psychological similarity in perception and cognition. Some examples are Miller and Nicely (1955), Luce (1963), Townsend (1971), Geyer and DeWald (1973) and examples with special emphasis on application of multiplicative confusion models are Falmagne (1972) and Wandmacher (1975). We will consider here this special type of confusion matrix (designated multiplicative confusion matrices). In these, $c_{ij}$ for $i \neq j$ can be factored into an element that is a function of $i$ and one that is a function of $j$. Since certain degenerate conditions arise when $n \leqslant 2$ (see e.g., Falmagne, 1972), it will henceforth be assumed that $n \geqslant 3$.

Most contemporary mathematical models of confusion matrices, including non-multiplicative models, can be captured by the following definition of confusion models.

DEFINITION 1. Let $S = \{S_i\}$ be a set of $n$ stimuli and $R = \{R_j\}$ be the set of $n$ corresponding responses, with $n \geqslant 3$. Then $M = \langle A, B, C, F \rangle$ is a confusion model for the system $\langle S, R \rangle$ in case $A$ and $B$ are real-valued parameter spaces of dimension $m_A$ *and* $m_B$, respectively, i.e.,

$$A \subseteq \mathbb{R}_e^{m_A}, \qquad B \subseteq \mathbb{R}_e^{m_B}$$

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for $m_A$, $m_B$ positive integers; $C$ is the set of all $n \times n$ row-stochastic (confusion) matrices, and $F$ is a function from $A \times B$ into $C$.

$F$ expresses the confusion cell entries in terms of the parameters. The reason for the two parameter spaces $A$ and $B$ is that it is often assumed that one set of parameters ($A$) relates only to experimental stimulus factors while the other ($B$) relates only to experimental response factors. For instance, the $A$ parameters might be functions of stimulus intensity or energy or pre- and poststimulus perceptual effects but the $B$ parameters might be functions of such experimental factors as relative payoff values or stimulus presentation probabilities. Of course, in general, the parameter spaces might be functions of yet other qualities but the stimulus vs. response factors play an important role in the following development.

The nontrivial import of Definition 1, when conjoined with the empirical interpretation of $A$ and $B$, is that not only can such a mathematical model typically be tested against (fitted to) a particular empirical confusion matrix, but the stimuli (and related factors such as intensity, etc.,) and responses (and related factors such as motivation or bias conditions) can be alternatively varied. Experimental variations of stimulus factors should produce variations only in parameters contained in $A$ and experimental manipulation of response-related factors( e.g., payoff conditions) should produce changes only in the $B$-parameters. Thus, a model can often be tested across experiments by observing the outcome of predicted parameter invariances or changes.

We now want to specialize the theoretical scope to multiplicative confusion models. There are several types of interest, which are convenient to group together in a single definition.

Definition 2. Let $M = \langle A, B, C, F \rangle$ be a confusion model for a system of $n$ stimuli and responses. Then $M$ is said to be multiplicative in case there are nonnegative functions $s_i(A \times B)$ and $r_j(A \times B)$, for $1 \leqslant i, j \leqslant n$, such that for all $i \neq j$, $[F(A \times B)]_{ij} = s_i(A \times B)\, r_j(A \times B)$, i.e., the $ij$th term in the confusion matrix given by $F(A \times B)$ can be expressed as a product of $s_i(\cdot)$ and $r_j(\cdot)$. A multiplicative confusion model is strong in case $s_i$ and $r_j$ depend only on $A$ and $B$, respectively. That is, a strong multiplicative confusion model is characterized by

$$[F(A \times B)]_{ij} = s_i(A)\, r_j(B),$$

for all $1 \leqslant i \neq j \leqslant n$. A multiplicative confusion model is restricted in case

$$s_i(\cdot) \sum_{k=1}^{n} r_k(\cdot) < 1,$$

for $1 \leqslant i \leqslant n$.

When a multiplicative model is not strong, then some $s_i$ or $r_j$ (or both) will be a function of both $A$ and $B$ and such a multiplicative model will be called weak. When a model is not restricted, then $s_i \sum_{k=1}^{n} r_k (1 \leqslant i \leqslant n)$ is not constrained to be less than 1 although it must be the case that $s_i \sum_{k \neq 1}^{n} r_k \leqslant 1$ $(1 \leqslant i \leqslant n)$ in order that the matrix be a confusion matrix as defined in Definition 1. Such a multiplicative model will be called general. It turns out that the weak–strong and the general–restricted distinctions are important in the model-equivalence relationships.

A multiplicative model will in general be characterized by its weak–strong and general–restricted description. For instance **SRMM** will refer to the class of strong restricted multiplicative models while **WGMM** will refer to the class of weak general multiplicative models. Similarly these abbreviations are also used as adjectives, so a WRMM is a weak restricted multiplicative model. Finally, the designation **CM** and CM will be employed to reference the full class of confusion models defined in Definition 1 and a member of this class, respectively.

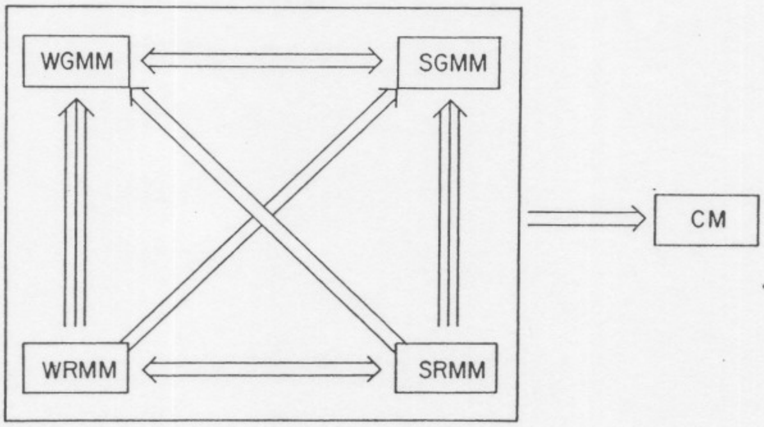
Now, when any two multiplicative confusion models $M_1$ and $M_2$ (with $n \geqslant 3$) predict the same numerical (empirical) confusion matrix, those models are related by a positive constant $K$, where $s_{iM_1} = Ks_{iM_2}$ and $r_{jM_1} = K^{-1}r_{jM_2}$ for $1 \leqslant i, j \leqslant n$ (e.g., see Falmagne 1972, Theorem 3). However, the restriction in models SRMM and WRMM is a constraint on the space of confusion models, since multiplication of one $s_i$ and division of the $r_j$ by $K$ obviously cannot alter the relationship of $s_i$ to $\sum_{k=1}^{n} r_k$ (it cancels out). The absolute size of $s_i$ is not affected by the restriction; rather it is the size of $s_i$ relative to the sum of the $r$'s that is important.

The usual notion of model equivalence prescribes that two models ($M_1$ and $M_2$) are equivalent if and only if each implies the other (e.g., see Greeno and Steiner, 1968; or Hurwicz, 1950). In the present investigation, whenever any confusion matrix predicted by $M_1$ (that is, any numerical realization produced by giving the parameters numerical values) can be predicted by $M_2$, without violation of the conditions or bounds on the parameters of $M_2$, $M_1$ is said to imply $M_2$. We will use the symbol "$\Rightarrow$" to stand for implication in one direction and "$\Leftrightarrow$" will refer to implication in both directions, that is, to model-equivalence.

It can be seen that any restricted model $\Rightarrow$ a general model since a general model can immediately be constructed with $s$'s and $r$'s identical to those of the restricted model over the appropriate range of $A$ and $B$, but can also possess $r$'s and $s$'s not obeying the constraint. Clearly, a general model cannot $\Rightarrow$ a restricted model.

Next, it is of interest here to consider the additional requirement that when the $r$'s or $s$'s of $M_1$ are functions of $A$ or $B$ only, as in the strong models in Definition 2, that the $r$'s and $s$'s of $M_2$ will also be functions only of $A$ or $B$, respectively. Implication or equivalence with this restriction included will be denoted "$\Rrightarrow$" and "$\Lleftarrow\!\!\Rrightarrow$" respectively. Thus, if $M_1 \Rrightarrow M_2$ (where both $M_1$ and $M_2$ are multiplicative) then both $M_1$ and $M_2$ are strong or both weak and implication in the usual sense holds as well. If $M_1 \Lleftarrow\!\!\Rrightarrow M_2$ then both are strong or both weak and they are additionally equivalent in the usual sense. It is clear that the present notions of equivalence can be immediately extended to classes of models so that, for example, **SRMM** $\Rrightarrow$ **WRMM** and **WGMM** $\Rightarrow$ **CM**, meaning that for any SRMM there exists an implied WRMM and for any WGMM (indeed, any multiplicative model whatever) there exists an implied CM.

Figure 1 summarizes these relations among the various classes of models. From the fact that **WRMM** $\Leftrightarrow$ **SRMM**, **WGMM** $\Leftrightarrow$ **SGMM**, **WRMM** $\Rrightarrow$ **WGMM** and **SRMM** $\Rrightarrow$ **SGMM**, we can conclude that **WRMM** $\Rightarrow$ **SGMM** and **SRMM** $\Rightarrow$ **WGMM** (the




CM = FULL CLASS OF CONFUSION MODELS
WGMM = CLASS OF WEAK GENERAL MULTIPLICATIVE MODELS
SGMM = CLASS OF STRONG GENERAL MULTIPLICATIVE MODELS
WRMM = CLASS OF WEAK RESTRICTED MULTIPLICATIVE MODELS
SRMM = CLASS OF STRONG RESTRICTED MULTIPLICATIVE MODELS
$\Longrightarrow$ = MODEL IMPLICATION
$\Rrightarrow$ = MODEL IMPLICATION PLUS BOTH MODELS ARE WEAK OR BOTH STRONG

FIG. 1. Summary of equivalence relations among the classes of multiplicative models and the full class of confusion models.

diagonal in Fig. 1), but it is false that **WRMM** $\Rightarrow$ **SGMM** or **SRMM** $\Rightarrow$ **WGMM** since the pairs differ on the strong–weak dimension.

The psychologically relevant empirical aspect of the strong vs. weak distinction is that in the strong models, the experimenter can in principle manipulate only the experimental bias factor (relative rewards on the various responses, etc.,) or only the sensory factor (stimulus intensity, etc.,) in which case only the $s_i$ terms or the $r_j$ terms, respectively, should change across the two conditions. For instance, high vs. low stimulus intensity with bias constant should affect only the $s$'s. The weak models in contrast predict that both the $r$'s and the $s$'s should be affected by stimulus intensity level.

Obviously, the predictions of the confusion matrix cells are on the 'true score' level and any particular experiment will result in empirical values that reflect one or more sources of variance and thus will not precisely equal the 'true score' value. For instance, suppose $c_{ij}$, $1 \leqslant j \leqslant n$ are the 'true scores' of row $i^0$ then clearly the distribution of the cell frequencies in an experiment involving the presentation of $N_i$ trials of stimulus $i$ will be multinomial. The empirical $c_{ij}$ will be estimates, $\hat{c}_{ij}$, of the true values. Similarly, predicted changes brought about by varying $A$ and $B$ parameters will be subject to statistical error.

The specific models on which we concentrate in this paper can now be presented in terms of their parameter spaces $A$ and $B$ and mappings from the parameter vectors to the confusion matrices included in $C$. Following their specification their equivalence relations with one another and with the models of Definition 2 will be given.

## Some Contemporary Multiplicative Models

The following models possess $nr$'s and $ns$'s, but each model allows parameter values that have the effect of multiplying the $s$'s by $K$ ($>0$) and dividing the $r$'s by the same $K$, without altering the confusion matrix values. There thus are only $2n - 1$ parameters in any of the models, since the $r$'s can be standardized to sum to 1 by transforming the original model with $K = \sum_{k=1}^{n} r_k$. In the case of each specific model, the indeterminacy involving exact values of $r$ and $s$ can be translated into similar indeterminacies of the respective parameters. Although all the present models have $2n - 1$ degrees-of-freedom in an algebraic sense, the restricted models and their equivalent counterparts constrain the legitimate values of the parameters in $A \times B$.

### *The All-or-None Model* (AON)

The All-or-None Model has received the most attention of the multiplicative models, in most part because of its clear connections with information processing notions of confusion and the related fact that it represents a limiting case of processing confusion models. The qualitative ideas behind the All-or-None Model are old, but recently it has been treated more quantitatively. It was referred to by Broadbent (1967) as the Pure Guessing Model and by Smith (1968) as the Pure Perceptibility Response Bias Model. It was developed from sensory activation notions, called the All-or-None Model, and compared with nonmultiplicative models in a 26-letter alphabetic confusion experiment by Townsend (1968, 1971). The All-or-None Model was significantly inferior to the more general models, which permitted the representation of stimulus similarity effects. It is also conceptually related to the Simple Fast Guess Model (Ollman, 1966; Yellott, 1971; Falmagne, 1972).

It is defined by $A \times B$, where $A = \langle p_i \rangle$ and $B = \langle b_j \rangle$, $1 \leqslant i, j \leqslant n$ and the $A \times B \to C$ mapping given by

$$\begin{aligned} c_{ij} &= (1 - p_i) b_j , & i \neq j, \\ &= p_i + (1 - p_i) b_i , & i = j, \end{aligned}$$

where $0 < p_i < 1$, $0 < b_j < 1$, and $\sum_{j=1}^{n} b_j = 1$.

The idea here is that the stimulus ($s_i$, $1 \leqslant i \leqslant n$) is perfectly recognized with probability $p_i$ (all information necessary for a correct response is processed) and hence response $R_i$ is made. But with probability $1 - p_i$, none of the information helpful toward making the correct response is processed and the observer must guess at random, responding $j$ with probability $b_j$ ($1 \leqslant j \leqslant n$). There are $n$ free $p$'s and $n - 1$ free $b$'s. It will be shown below that AON is a member of **SRMM**.

The rest of the models we consider bear close affinity with the Choice Model of confusion (Luce, 1963), all the $c_{ij}$ involving ratios of strengths of potential responses.

*The Broadbent Response Bias Model* (BRB)

This model is similar in form to Luce's model for weight discrimination (1959, p. 31) and was discussed by Broadbent in the context of word recognition (1967). Here, $A = \langle \alpha_i \rangle$ and $B = \langle V_j \rangle$, $1 \leqslant i, j \leqslant n$, and

$$c_{ij} = \frac{V_j}{\alpha_i V_i + \sum_{k \neq 1}^{n} V_k}, \qquad i \neq j,$$

$$= \frac{\alpha_i V_i}{\alpha_i V_i + \sum_{k \neq 1}^{n} V_k}, \qquad i = j,$$

where it is supposed that $1 < \alpha_i < +\infty$ and $0 < V_j < +\infty$. The interpretation of $1 < \alpha_i$ is that the effect of presenting a stimulus tends to increase the strength of the respective response ($\alpha_i V_i > V_i$). The probability of $c_{ij}$ is then the ratio of the strength of alternative $j$ relative to the total sum of strengths applicable when stimulus $s_i$ is presented. There are $n$ free $\alpha$'s and $n - 1$ free $V$'s since the latter can be normalized to sum to 1 by dividing the numerator and denominator of $c_{ij}$ $(i \neq j)$ by $K = \sum_{k=1}^{n} V_k$. It is an WRMM.

*The Multiplicative Similarity Model* (MSM)

This model was suggested by Smith (1968) in an investigation of the relationship of Luce's Choice Model (1963) to the All-or-None Model. It is defined by $A = \langle a_i \rangle$, $B = \langle c_j \rangle$, and

$$c_{ij} = \frac{a_i a_j c_j}{\sum_{k \neq i}^{n} a_i a_k c_k + c_i}, \qquad i \neq j,$$

$$= \frac{c_i}{\sum_{k \neq i}^{n} a_i a_k c_k + c_i}, \qquad i = j.$$

It derives from Luce's 1963 Choice-confusion model by assuming that the similarity parameters $\eta_{ij} = a_i \cdot a_j$, that is, stimulus similarity can be decomposed into a product of contributions from the two stimuli. Although the stipulation that $\eta_{ii} = 1$ and $\eta_{ij} < 1$, $i \neq j$, (two objects cannot be more similar than an object is to itself) can be met by only assuming that $n - 1$ of the $n$ $a$-parameters have values less than or equal to 1, it seems natural to assume $0 < a_i < 1$ for all $1 \leqslant i \leqslant n$ and we shall do so. The biases, the $c_j$, have the same meaning as the $\beta_j$ in the 1963 formulation, with $0 < c_j < +\infty$. There are $n$ free $a$'s and $n - 1$ free $c$'s in the usual formulation of the model, and we shall learn that it is contained in the class **WRMM**.

*The Simply Biased Choice Model* (SBC)

Derived by Falmagne (1972), this model possesses parameter sets $(g_i)$ and $(f_j)$ where the $g$'s are to be interpreted as the total strength of the presented stimulus and the $f$'s are strengths of unpresented stimuli ($1 \leqslant i, j \leqslant n$). Although the $g$'s must certainly contain some effect of the stimulus and the $f$'s some effect of response bias, it may be open to speculation whether the $g$'s should also reflect response bias (toward the correct response). Since $g_i$ has not been written explicitly as a function of both stimulus and bias factors (as contrasted with, say, $\alpha_i V_i$ as the strength of the presented stimulus in BRB), we shall take the tack of assuming that $A = \langle g_i \rangle$ and $B = \langle f_j \rangle$. The appropriate mapping to the confusion matrix is

$$c_{ij} = \frac{f_j}{\sum_{k \neq i}^{n} f_k + g_i}, \qquad i \neq j,$$

$$= \frac{g_i}{\sum_{k \neq i}^{n} f_k + g_i}, \qquad i = j,$$

where $0 < f_j, g_i < +\infty$. We shall also assume that $g_i > f_i$, that is the strength of the $i$th alternative is greater when it has been presented than when it has not. There are $n$ free $g$'s and $n - 1$ free $f$'s, where (as with the $V$'s in BRB and the $c$'s in MSM) the $f$'s can be normalized to add to 1. As are the previous two models, SBC will be shown to be a WRMM.

## Equivalence Results and Discussion [2]

### *Equivalence Relations*

Figure 2 presents the equivalence and implication relations among the various models, and shows the classes of models to which they belong (**SRMM**, **WRMM**, etc.,). The 'psychologically intuitive' models listed to the left at the bottom of Fig. 2 are those defined in the preceding section. The general models on the right are formed by relaxation of the parameter constraints in the original models. The models within a box in Fig. 2 are both model-equivalent and identical on the weak–strong dimension. Thus, the AON model belongs to the class SRMM, $\Rightarrow$ a generalized version of itself (AON′), and is $\Leftrightarrow$ to the other multiplicative models given in the previous section (BRB, MSM and SBC).

The relationships of Fig. 2 can be easily verified by reference to Table 1, which gives the equivalence mappings between the specific models, and by reference to Fig. 1 and Definition 2. Table 1 gives the mappings as: (parameters of the model given by row) $\rightarrow$ (parameters of the model given by column) in the off-diagonal cells. The diagonal cells of Table 1 show the kind of model each is and the mappings between the specific model and the $s$'s and $r$'s of an appropriate multiplicative model of Definition 2.

It can thus be confirmed that all the present models presented previously meet the restriction that $s_i \sum_{k=1}^{n} r_k < 1$ and that for any set of $s$'s and $r$'s of the restricted type

[2] Smith (1968) was responsible for giving the parameter mappings between AON and MSM and Nakatani (1970) did the same for BRB and AON. The other mappings have not been previously exhibited, nor has the weak vs. strong distinction been raised. The potential importance of bounds on the $s_i$ relative to the $r_j$ ($s_i < (\sum_{k=1}^{n} r_k)^{-1}$, $1 \leqslant i, j \leqslant n$) in testing multiplicative models has also not been reported or dealt with earlier.

TABLE 1

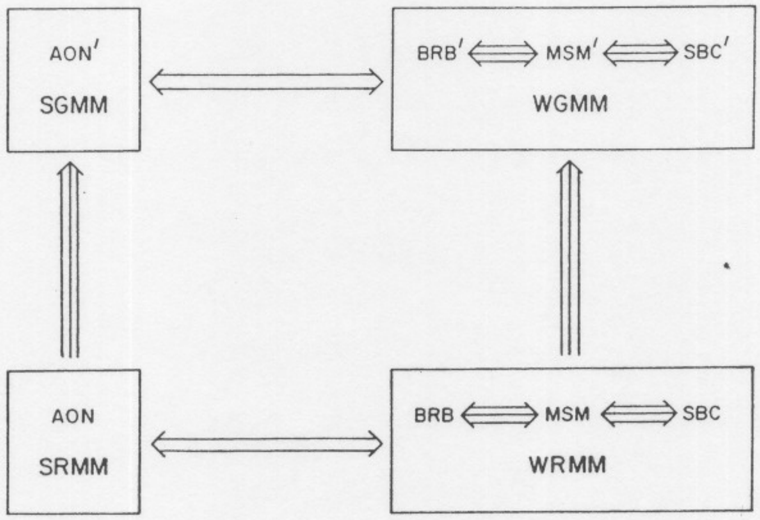
Equivalence Mapping among Multiplicative Models[a,b]

| | AON | BRB | MSM | SBC |
|---|---|---|---|---|
| AON | AON ∈ **SRMM**<br>$s_i = 1 - p_i$<br>$r_j = b_j$<br>same as above | $\alpha_i = \dfrac{p_i + (1 - p_i)b_i}{(1 - p_i)b_i}$<br>$v_j = b_j$ | $a_i = \left[\dfrac{(1 - p_i)b_i}{p_i + (1 - p_i)b_i}\right]^{1/2}$<br>$c_j = \left[\dfrac{b_j(p_j + (1 - p_j)b_j)}{1 - p_j}\right]^{1/2}$ | $g_i = \dfrac{p_i + (1 - p_i)b_i}{1 - p_i}$<br>$f_j = b_j$ |
| BRB | $1 - p_i = \dfrac{1}{\alpha_i V_i + \sum_{k \neq i}^{n} V_k}$<br>$b_j = V_j$ | BRB ∈ **WRMM**<br>$s_i = \dfrac{1}{\alpha_i V_i + \sum_{k \neq i}^{n} V_k}$<br>$r_j = V_j$<br>$\alpha_i = \dfrac{1 - s_i \sum_{k \neq i}^{n} r_k}{s_i r_i}$ | $a_i = (\alpha_i)^{-1/2}$<br>$c_j = V_j(\alpha_j)^{1/2}$ | $g_i \doteq \alpha_i V_i$<br>$f_j = V_j$ |

| | | | | |
|---|---|---|---|---|
| | | | MSM $\in$ **WRMM** | |
| MSM | $1 - p_i = \dfrac{a_i}{a_i \sum_{k\neq i}^{n} a_k c_k + c_i}$<br>$b_j = a_j c_j$ | $\alpha_i = \dfrac{1}{a_i^2}$<br>$V_j = a_j c_j$ | $s_i = \dfrac{a_i}{a_i \sum_{k\neq i}^{n} a_k c_k + c_i}$<br>$r_j = a_j c_j$<br>$a_i = \left[\dfrac{s_i r_i}{1 - s_i \sum_{k\neq i}^{n} r_k}\right]^{1/2}$<br>$c_j = r_j \left[\dfrac{1 - s_j \sum_{k\neq j}^{n} r_k}{s_j r_j}\right]^{1/2}$ | $g_i = \dfrac{c_i}{a_i}$<br>$f_j = a_j c_j$ |
| | | | | SBC $\in$ **WRMM** |
| SBC | $1 - p_i = \dfrac{1}{g_i + \sum_{k\neq i}^{n} f_k}$<br>$b_j = f_j$ | $\alpha_i = \dfrac{g_i}{f_i}$<br>$V_j = f_j$ | $a_i = (f_i/g_i)^{1/2}$<br>$c_j = (f_j g_j)^{1/2}$ | $s_i = \dfrac{1}{g_i + \sum_{k\neq i}^{n} f_k}$<br>$r_j = f_j$<br>$g_i = \dfrac{1 - s_i \sum_{k\neq i}^{n} r_k}{s_i}$ |

[a] AON = All-or-None Model, BRB = Broadbent Response Bias Model, MSM = Multiplicative Similarity Model, SBC = Simply Biased Choice Model.

[b] For further explanation, refer to the text and Fig. 2.




PARAMETER CONSTRAINTS

| RESTRICTED, PSYCHOLOGICALLY INTUITIVE MODELS | GENERAL MODELS |
|---|---|
| AON: $1 > p_i > 0$, $1 > b_j > 0$ | AON′: $1 > p_i > -\infty$, $1 > b_j > 0$ |
| BRB: $+\infty > \alpha_i > 1$, $+\infty > V_j > 0$ | BRB′: $+\infty > \alpha_i > 0$, $+\infty > V_j > 0$ |
| MSM: $1 > a_i > 0$, $+\infty > c_j > 0$ | MSM′: $+\infty > a_i > 0$, $+\infty > c_j > 0$ |
| SBC: $+\infty > g_i > f_i > 0$ | SBC′: $+\infty > g_i > 0$, $+\infty > f_j > 0$ |

FIG. 2. Summary of relations of restricted and general multiplicative models with each other and the major multiplicative classes.

of model, there is a version of AON, BRB, MSM, and SBC, respectively, which is equivalent. Note that although BRB, MSM, and SBC can possess one or more $s_i > 1$ for a given confusion matrix, this does not enlarge their scope over that of AON since for any confusion matrix predicted by one of these, an identical one can be predicted with other parameter values resulting in $s_i < 1$ for all $1 \leqslant i \leqslant n$. In fact, it is interesting that the same transformation resulting in $\sum_{k=1}^{n} r_k = 1$ for each of the models results in $s_i < 1$. This transformation is $s_i' = Ks_i$ and $r_j' = r_j/K$, where $K = \sum_{k=1}^{n} r_k$ , $1 \leqslant i \leqslant n$, and the $s$'s and $r$'s are written in terms of the parameters of the respective models as given in Table 1. For BRB and SBC, this is the same transformation that normalizes the bias parameters ($K = \sum_{k=1}^{n} V_k$ and $K = \sum_{k=1}^{n} f_k$ , respectively) but in MSM, $K = \sum_{k=1}^{n} a_k c_k$ .

As noted above, Fig. 2 also shows what happens when the parameter constraints of the present psychological models are relaxed. The resulting general multiplicative models are contained in **WGMM** or **SGMM** since the restriction $s_i \sum_{k=1}^{n} r_k < 1$ is no longer in force.[3]

[3] There is a fairly unusual circumstance in which there may be an intuitive rationale for extending the parameter spaces as is done in the general multiplicative models. In degraded displays (e.g., brief and noisly) it is possible that the percept of $S_i$ could be more similar to $S_j$ than to itself. For instance, in an alphabetic confusion experiment, presentation of an $E$ may result in the percept of $F$ due to loss of the lower horizontal feature, whereas presentation of the $F$ leads to something

Table 2 exhibits an example numerical confusion matrix which cannot be predicted by the restricted models. It was generated by setting $s_1 = 1.26$, $s_2 = 0.50$, $s_3 = 0.40$; $r_1 = 0.67$, $r_2 = 0.165$, $r_3 = 0.165$. Obviously, $s_1 \sum_{k=1}^{n} r_k = 1.26 > 1$ thus violating the restriction of AON, BRB, MSM, and SBC. The generalized models can, of course, predict this matrix. Thus in BRB′, $\alpha_1 = [1 - s_1(r_2 + r_3)]/s_1 r_1 = [1 - 1.26(0.165 + 0.165)]/(1.26 \times 0.67) = 0.69 < 1$, which is not an acceptable parameter value for BRB, but is for BRB′.

TABLE 2

Example of a Confusion Matrix not Predictable by AON, BRB, MSM, or SBC

| | $R$ | | |
|---|---|---|---|
| $S$ | 1 | 2 | 3 |
| 1 | 0.58 | 0.21 | 0.21 |
| 2 | 0.33 | 0.59 | 0.08 |
| 3 | 0.27 | 0.07 | 0.66 |

It should be mentioned that alternative interpretations exist of the bias parameters in AON, the $b_j$'s, which transform that model into a WRMM. Wandmacher (1977) expresses $b_j$ (in the present notation) as

$$b_j = \frac{(1 - p_j)\, h_j v_j}{\sum_{k=1}^{n} (1 - p_k)\, h_k v_k} \qquad (1 \leqslant j \leqslant n),$$

where $h_k$ equals the subject's prior probability that stimulus $S_k$ will be presented and $v_k$ is a response strength parameter assumed to be independent of the a priori probability of $S_k$ $(1 \leqslant k \leqslant n)$. If $h_j$ equals the a priori probability of $S_j$ and $v_i = v_j$ for all $i$, then this expression can be seen to be the a posteriori probability that stimulus $S_j$ was presented, given one is in the no-information state. Here, because the $b$'s are themselves functions of the $p$'s, $A = (p_i)$, $B = (h_j, v_j)$, $1 \leqslant i, j \leqslant n$, and $F_{ij}(A, B) = s_i(A)\, r_j(A, B)$ and the model then belongs to **WRMM**. This interpretation will not be further considered here but the reader may refer to Wandmacher (1977) for more detail.

like an $I$ because of loss of its two horizontal features. In this instance presentation of the stimulus $F$ leads to a weaker response strength towards $F$ than towards $I$ and a weaker tendency to say "$E$" when it is presented than to report an $F$. Here, the WGMM models BRB′, MSM′ and SBC′ (with extended parameter values) might provide a more adequate account of the data than the restricted models. However, AON′ still has no intuitive justification because the concept of '$p$' as representing a probability is violated.

*Implications for Model Testing*

The remainder of the paper is addressed to some points concerning the experimental testing of the presently considered confusion models. We do not emphasize questions of estimation and model fitting per se although maximum likelihood estimation is straightforward because of the assumed multiplicativity (Smith, 1968) and method-of-moment estimates were given by Townsend (1971), and either of these is easily extended to the other equivalent models via the mappings of Table 1. The present remarks will rather be directed to interesting aspects of the models' structures.

Any multiplicative model can obviously be falsified in principle by a single (experimentally derived) empirical confusion matrix. For instance, the off-diagonal cells should evidence row $\times$ column independence as tested by an appropriate $\chi^2$ test of independence. On the other hand, any pair of models associated by model equivalence ($\Leftrightarrow$ or $\not\Leftrightarrow$) cannot be tested against one another by a single empirical confusion matrix. If an empirical matrix turns out to be multiplicative with $s_i \sum_{k=1}^{n} r_k < 1$ then the restricted class of models is supported, of which all the present 'psychologically intuitive' restricted models are members. But, if $s_i \sum_{k=1}^{n} r_k > 1$ then that class is tentatively falsified with respect to the general class of models (assuming of course adequate data, estimation, and statistical tests).

The strong vs. weak issue cannot be tested with a single empirical matrix, but can be tested across experimental conditions. In fact, it is important to note that when the sensory and bias conditions are varied across experimental conditions, even models within the same strong vs. weak as well as restricted vs. general class can be tested against one another. As an example, suppose two values of stimulus intensity were employed, while response factors were held unchanged, thus yielding two empirical confusion matrices. Assume that the postulate (following Definition 1) that this sensory manipulation affects only the parameter set $A$ is correct. Suppose further that the results suggested multiplicativity ($S_i \times R_j$ off-diagonal entries were independent within each single matrix) but that examination of the matrices across alterations of the stimulus intensity factor showed that both the $s$'s as well as the $r$'s were changing as a function of this factor. Assume finally that estimates of the $s$'s and $r$'s supported $s_i < \sum_{k=1}^{n} r_k$ for $1 \leqslant i \leqslant n$. Then the investigator should look at models contained in **WRMM** for candidates for the 'true' model. The specific models of the present investigation meeting these terms are BRB, MSM, and SBC. AON is ruled out because it predicts that only the $s$'s should change with stimulus intensity and the general models need not be considered since the restriction of **SRMM** and **WRMM** is not violated. Finally, since the three candidates (BRB, MSM, SBC) differ according to the ways in which the $s_i$ depend on $(A, B)$, the correct model should evidence alterations primarily in the estimates of $A$ whereas the incorrect models should tend to yield alterations both in $B$ as well as $A$.

Method-of-moment estimates of the AON parameters were given in Townsend (1971). These basically involved finding inverse mappings carrying the $c_{ij}$ back to the (presumed) parameters. The estimates depend only on multiplicativity and not on whether the models are general or restricted. For example, suppose $n = 3$ and the $r$'s are taken to satisfy $\sum_{k=1}^{n} r_k = 1$ (from the above development, we know this involves no loss of generality). Then using the bias estimates from Townsend (1971), it follows that (for instance) $\hat{r}_2 = 1/(1 + \hat{c}_{13}/\hat{c}_{12} + \hat{c}_{31}/\hat{c}_{32})$ and then, from $c_{12} = s_1 r_2$, an estimate of $s_1$ is $\hat{s}_1 = (1 + \hat{c}_{13}/\hat{c}_{12} + \hat{c}_{31}/\hat{c}_{32})\hat{c}_{12}$. Clearly, $s_1$ is not constrained to be less than 1 and when $\hat{s}_1 > 1$, evidence will tend to accrue against the restricted models. Employment of these estimators with the example of Table 2 above, results in retrieval of the 'general' parameters; in particular $\hat{s}_1 = 1.26$.

In a preliminary search for multiplicativity, the investigator may check for certain nonparametric relations (not all of which may be initially independent). Some of these are obvious, such as the predicted constancy of $c_{ik}/c_{jk}$ over $k$ and of $c_{ij}/c_{ik}$ over $i$, and $c_{ij}c_{jk}c_{ki}/c_{ik}c_{kj}c_{ji} = 1$ $(1 \leqslant i, j, k \leqslant n$; and $i \neq j \neq k)$. Others can be found by noting that several functions of the empirical confusion matrix entries should yield the same values of a particular $s$ or $r$, if the appropriate model is multiplicative. Thus,

$$\begin{aligned}
\hat{s}_1 &\simeq (1 + \hat{c}_{12}/\hat{c}_{12} + \hat{c}_{31}/\hat{c}_{32})\hat{c}_{12} \\
&\simeq (1 + \hat{c}_{12}/\hat{c}_{13} + \hat{c}_{21}/\hat{c}_{23})\hat{c}_{13} \\
&\simeq 1 - \{\hat{c}_{11} - (1 - \hat{c}_{11})[1/(\hat{c}_{23}/\hat{c}_{21} + \hat{c}_{32}/\hat{c}_{31})] \\
&\qquad - \hat{c}_{12}[1 + \hat{c}_{13}/\hat{c}_{12} + \hat{c}_{31}/\hat{c}_{32}] - \hat{c}_{13}[1 + \hat{c}_{12}/\hat{c}_{13} + \hat{c}_{21}/\hat{c}_{23}] + 2\}\tfrac{1}{3},
\end{aligned}$$

where the last expression comes from Townsend (1971) and the first two follow immediately from multiplicativity and estimates of the biases. The point is that these three different right-hand sides of the equation, or estimators of $s_1$, should all be approximately equal if the data can be explained by a multiplicative model. To be sure, no precise statistical tests based on these equalities have yet been devised. It would probably be wise to combine inspection of the nonparametric relations with more traditional, but sometimes less informative analyses of the matrix structure (e.g., STEPIT).

It may finally be observed that at this time the only available multiplicative models that seem generally intuitively reasonable are of the restricted type (**WRMM** or **SRMM**).

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